11 Polygons and 3-D shapes

1 Triangles

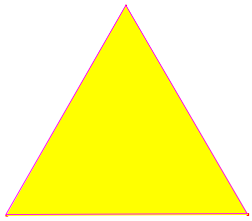
A triangle is a three-sided polygon. Sides of a polygon are also called edges.

Triangles can be classified either by sides or by angles.

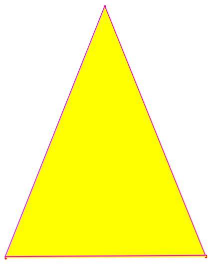
By sides:

Equilateral Triangle

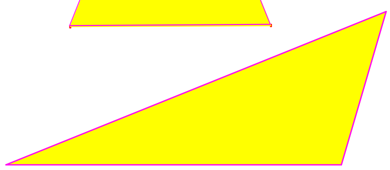
It is a triangle that has:
Three equal sides
Three equal angles of 60 degrees.



Isosceles Triangle
It is a triangle that has:
Two sides of equal length.
Two equal angles

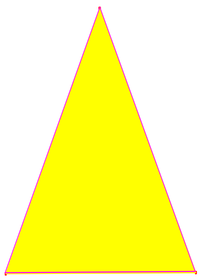


Scalene Triangle
It is a triangle that has:
Three sides of different lengths.
Three different angles

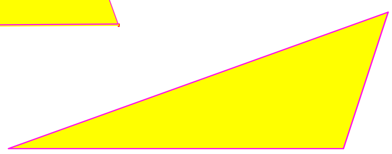


By angles:

Acute Triangle
It is a triangle that has three acute angles.

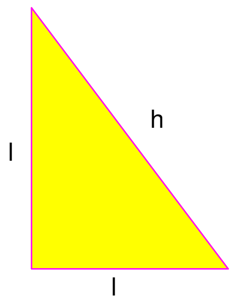


Obtuse Triangle
It is a triangle that has an obtuse angle, (measures more than 90º).

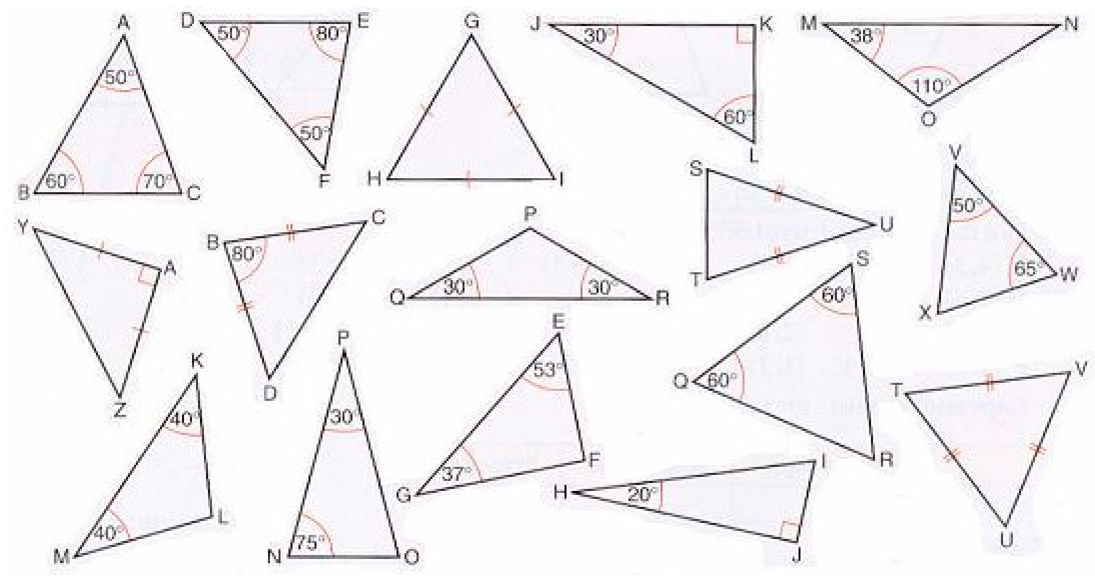


Right Triangle
It is a triangle that has a right angle.

The side opposite the right angle is called the **hypotenuse**. The two sides that form the right angle are called the **legs**.



Exercise 1 Classify the following triangles by their sides and by their angles. Find the asked angles of the triangles.



Triangle	By the sides	By the angles	Calculate
ABC			
DEF			
HGI			$\hat{H}$ =
JKL			
MNO			$\hat{N}$ =
YAZ			$\hat{Y}$ = $\hat{Z}$ =
BCD			$\hat{C}$ = $\hat{D}$ =
QPR			$\hat{P}$ =
SUT			
XVW			$\hat{X}$ =
MKL			$\hat{L}$ =
NPO			$\hat{O}$ =
GEF			$\hat{F}$ =
QSR			$\hat{R}$ =
UTV			$\hat{U}$ = $\hat{T}$ = $\hat{V}$ =
HIJ			$\hat{I}$ =

2 Equality of triangles

Two triangles are equal if the angles and the sides of one of them are equal to the corresponding angles and sides of the other

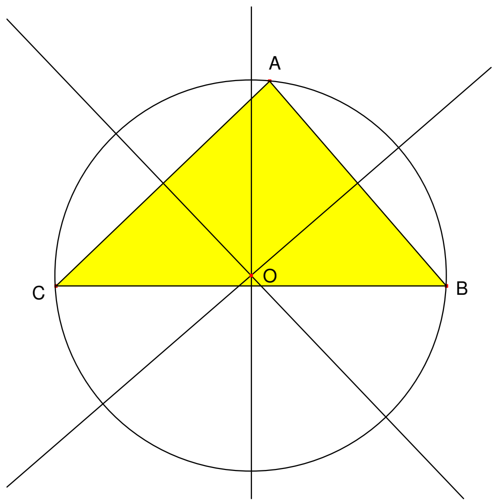
But we can assure that two triangles are equal if:

I. The lengths of their corresponding sides are equal
II. Two corresponding sides of two triangles and their included angles are equal.
III. One side and the angles at any side are equal.

3 Points and lines associated with a triangle

3.1 Perpendicular bisectors, circumcentre

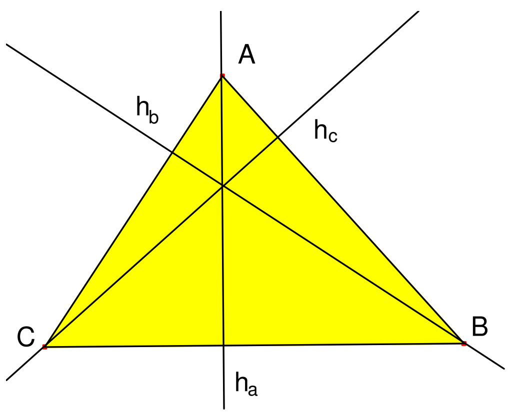
A perpendicular bisector of a triangle is a straight line passing through the midpoint of a side and perpendicular to it.



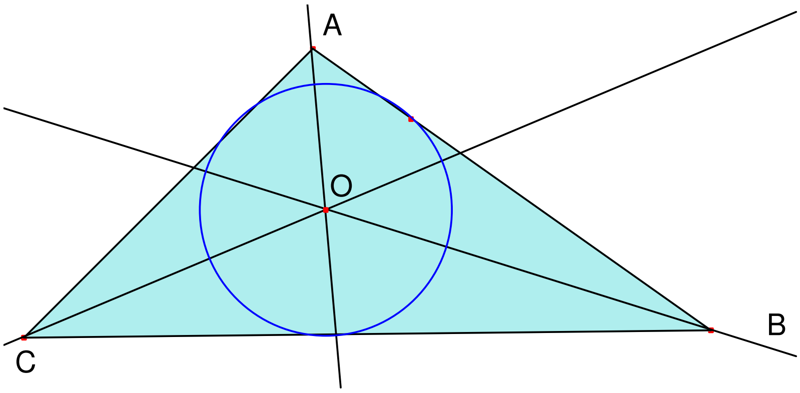
The three perpendicular bisectors meet in a single point, it is called the triangle's circumcentre; this point is the centre of the circumcircle, the circle passing through the three vertices.

3.2 Altitudes, orthocentre

An altitude of a triangle is a straight line through a vertex and perpendicular to the opposite side. This opposite side is called the *base* of the altitude, and the point where the altitude intersects the base (or its extension) is called the *foot* of the altitude.
The length of the altitude is the distance between the base and the vertex. The three altitudes intersect in a single point, called the orthocentre of the triangle.

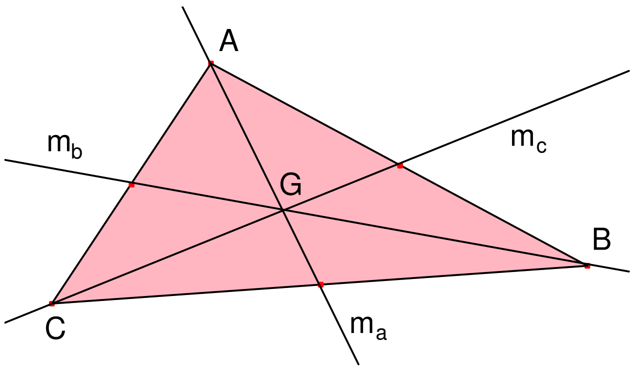


3.3 Angle bisectors, incentre



An angle bisector of a triangle is a straight line through a vertex, which cuts the corresponding angle in half. The three angle bisectors intersect in a single point called the incentre, which is the centre of the triangle's incircle.
The incircle is the circle, which lies inside the triangle and is tangent to the three sides.

3.4 Medians, barycentre



A median of a triangle is a straight line through a vertex and the midpoint of the opposite side.

The three medians intersect in a single point, the triangle's barycentre.

This is also the triangle's centre of gravity.

Exercise 2

a) **Construct a triangle with a base of 6 cm and angles of 60º and 45º.**

b) **Draw the three perpendicular bisectors and check that the three of them have a common point. Which is its name?**

c) **Use a compass to draw the circumcircle.**

Exercise 3

a) **Construct a triangle with a base of 5 cm, one side 7 cm and the angle between these sides of 50º.**

b) **Draw the three altitudes and check that the three of them have a common point. Which is its name?**

Exercise 4

a) **Construct a triangle with sides of 9.5 cm, 7.2 cm and 6 cm.**

b) **Draw the three angle bisectors and check that the three of them have a common point. Which is its name?**

c) **Draw the incircle of the triangle**

Exercise 5

a) **Construct a triangle with sides of 6.4 cm, 7.8 cm and the angle between these sides of 37º.**

b) **Draw the three medians and check that the three of them have a common point. Which is its name?**

c) **Check that the barycentre in every median is at a distance of the vertex double than the one it has to the midpoint of the side.**

Exercise 6 Draw in your notebook each of these triangles and find the asked point.

Triangle	DATA (lengths in cm)	POINT
ABC	a = 5 cm, b = 3 cm, c = 6 cm	Orthocentre
DEF	d = 6 cm, $\hat{E} = 33º$, $\hat{F} = 52º$	Barycentre
GHI	h = 7 cm $\hat{H} = 53º$ $\hat{I} = 67º$	Incentre
JKL	k = 4 cm, j = 6 cm $\hat{L} = 42º$	Circumcentre
MNO	m = 8 cm n = 6.6 cm $\hat{M} = 62º$	Circumcentre
PQR	p = 6 cm, q = 8 cm, r = 10 cm	Orthocentre and circumcentre

4 The Pythagorean Theorem

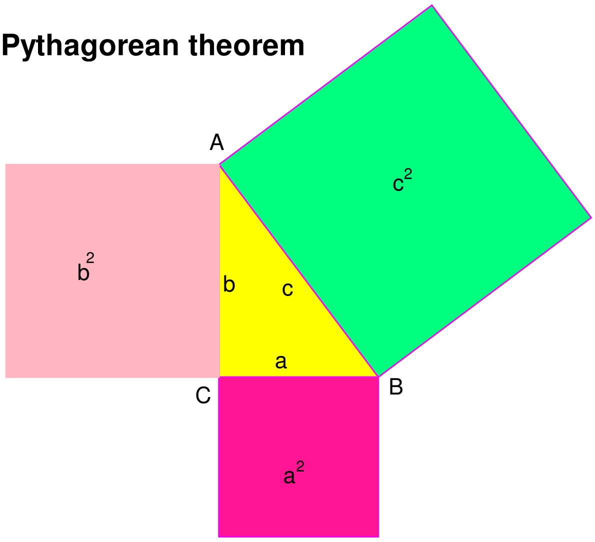
In a right triangle one of the angles of the triangle measures 90 degrees. The side opposite the right angle is called the hypotenuse. The two sides that form the right angle are called the legs.

In a right triangle the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse. This is known as the Pythagorean Theorem.

For the right triangle in the figure, the lengths of the legs are *a* and *b*, and the hypotenuse has length *c*. Using the Pythagorean Theorem, we can write that $a^2 + b^2 = c^2$.

The acute angles of a right triangle are complementary.

Pythagorean theorem



Exercise 7 Find the value of the third side of the following right triangles ABC in which $\hat{A} = 90º$. Round the calculations to the nearest hundredth:

a) **b = 7 cm, a = 9 cm.**

b) **b = 3 cm , c = 4 cm**

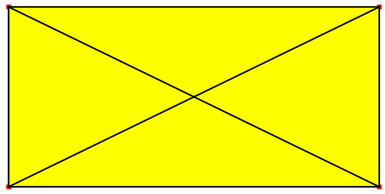
c) **a = 12.5 cm, c = 8.6 cm**

d) **a = 0.34 cm, c = 0. 27 cm**

5 Quadrilaterals

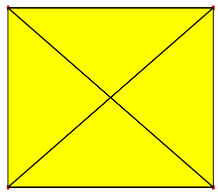
A quadrilateral is a polygon with four sides. The four angles of any quadrilateral add up to 360º.

These are the special quadrilaterals



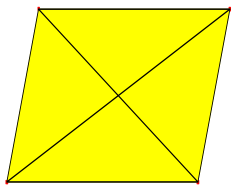
- Rectangle

All the angles are right angles.
Opposite sides are equal.
Diagonals have the same length and bisect each other.



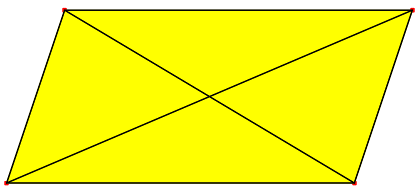
- Square

All the angles are of 90º
All the sides are equal in length
Diagonals have the same length and bisect each other at right angles.



- Rhombus

All the sides have the same length.
Opposite sides are parallel.
Opposite angles are equal.
Diagonals bisect each other at right angles.



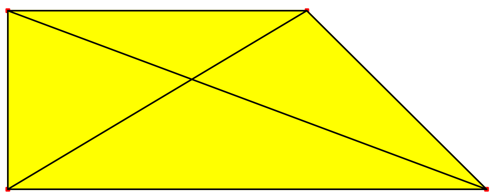
- Parallelogram or rhomboid

Opposite sides are parallel and have the same length.
Opposite angles are equal.
Diagonals have the same length and bisect each other.

- Trapezium (U.K.) or

Trapezoid (U.S.A.)

One pair of sides is parallel. The two sides that are parallel are called the bases of the trapezium.

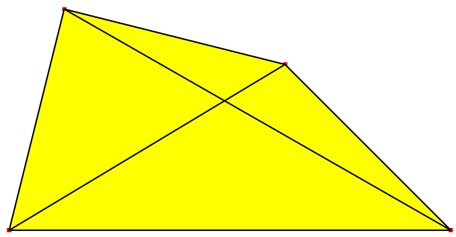


Trapezium (U.K.)

In an isosceles trapezium the base angles are equal, and so is the pair of non-parallel sides.

- Trapezoid (U.K.) or Trapezium (U.S.A.)

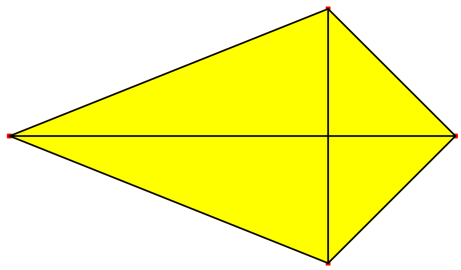
Quadrilateral that has no parallel sides



Trapezoid (U.K.)

- Kite

It is a special type of trapezoid in which:
The adjacent sides are equal.
Diagonals intersect each other at right angles



Exercise 8 One of the sides of a rectangle measures 4 cm and the diagonal 6 cm

a) Construct the rectangle.
b) Calculate the other side and the perimeter.

Exercise 9 In a rectangle the length of its sides are 8.3 cm and 5.4 cm. Calculate the length of its diagonal.

Exercise 10 Construct a rhombus with diagonals 4 cm and 7 cm. Calculate the perimeter.

Exercise 11 Construct a square with a diagonal of 7.5 cm. Calculate the perimeter.

Exercise 12 Say how many axes of symmetry there are in:

a) **A square**

b) **A rectangle**

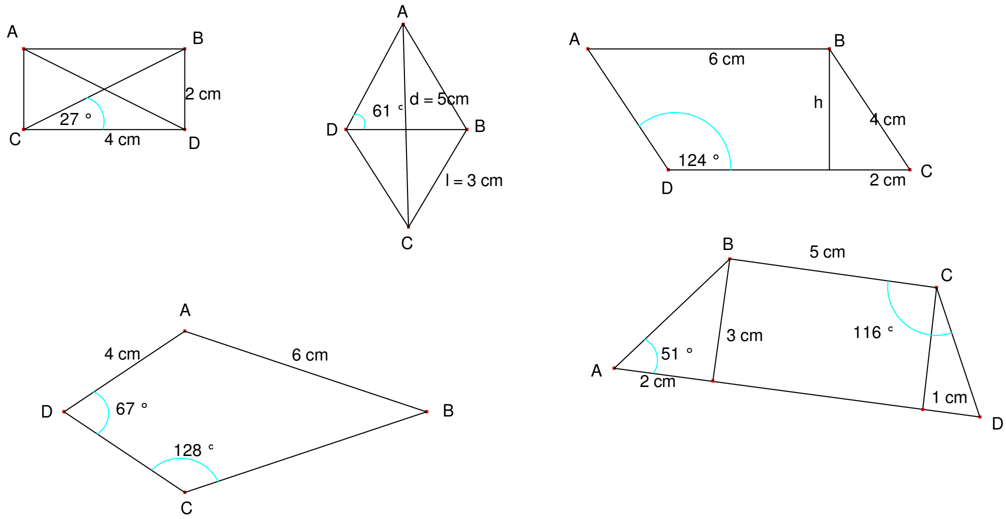
c) **A rhombus**

d) **A trapezium**

e) **A pentagon**

f) An hexagon

Exercise 13 Find all the missing sides and angles in the polygons below.

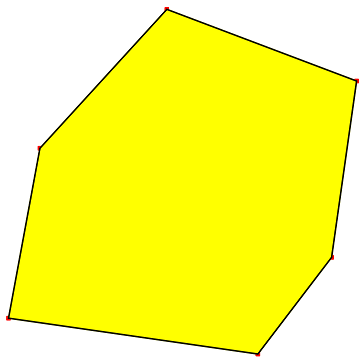


6 Regular polygons

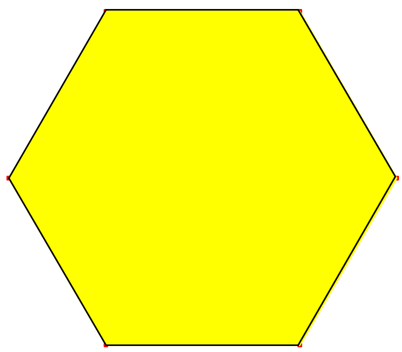
6.1 Definition and names

A **regular polygon** is a polygon in which all angles are equal and all sides have the same length.

Others polygons are called **irregular polygons**



Irregular hexagon



Regular Hexagon

A five-sided polygon is a **Pentagon**

A six-sided polygon is a **Hexagon**

A seven-sided polygon is a **Heptagon**

An eight-sided polygon is an **Octagon**

A nine-sided polygon is a **Nonagon**

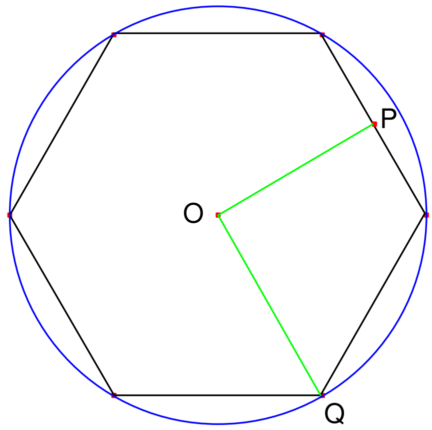
A ten-sided polygon is a **decagon**

With more than ten sides it is better to call them a polygon of **"n"** sides.

6.2 Proprieties of the regular polygons

All the regular polygons can be circumscribed by a circle, this is called the **circumcircle**, and it is a circle which contains all the vertices of the polygon. The centre of this circle is called the **circumcenter** and it is the **centre** of the regular polygon. The radius of this circle is also the **radius** of the polygon.

Apothem is the line drawn from the centre of the polygon perpendicular to a side.



Example:

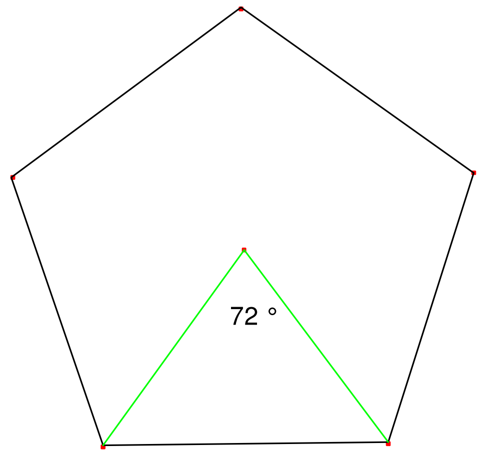
In this hexagon

O is the centre

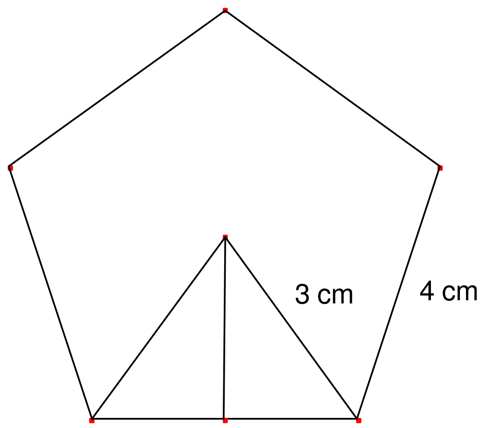
OP is the apothem

OQ is the radius

Central angle is the angle formed with two radiuses drawn in two consecutive vertices, it is equal to 360º divided by the number of sides



Exercise 14 Calculate the apothem of this pentagon



Exercise 15 Construct a regular hexagon with a radius of 3 cm. Calculate the apothem and the central angle

7 Circle and circumference

7.1 Definitions

- A circumference is the collection of points in a plane that are all at the same distance from a fixed point. The fixed point is called the **centre**.

- A circle is the shape inside the circumference.

- Circle is commonly used meaning circumference

7.2 Lines in a circle

1. **Radius** is the distance from the centre to the edge. (Segment OP)

2. **Diameter** is the segment between two points of the circle that passes through the centre. (Segment QR)

3. **Chord** is a straight line between two points of the circumference. (Segment CD)

4. **Arc** is a part of the circumference of a circle. (Curve CD)

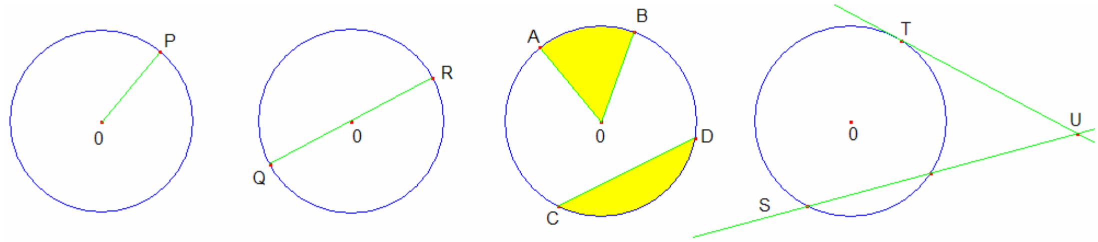
5. **Sector** is the shape made by two radiuses (radii) of the circle. (OAB)

6. **Segment** is a shape made by a chord and an arc. (Curve CD and segment CD)

7. **Tangent** is a straight line that touches the circle at only one point. (TU)

A tangent of a circle is always perpendicular to a correspondent radius

8. **Secant** is a line that intersects two points of a circle. (SU)



7.3 Positions of circles

- **Concentric circles** are circles, which have the same centre
- **Eccentric circles** have different centres.
- **Interior circles** are circles, one inside the other. They can be concentric or eccentric circles.
- **Exterior circles** have their centres at a distance greater or equal to the sum of their radiuses.
- **Tangent circles** have a common point. They can be interior or exterior circles.
- **Secant circles** have two common points.

Exercise 16 Draw circles in your notebook in all the different positions that you can see in 7.3

Exercise 17

a) **Construct a circumference with a radius of 5 cm and a chord of 7cm.**

b) **b) Calculate the distance from the centre to the chord.**

c) **Draw two secants and one tangent to the circle.**

3-D shapes.

8 Polyhedrons

Polyhedrons are geometric solids whose faces are formed by polygons

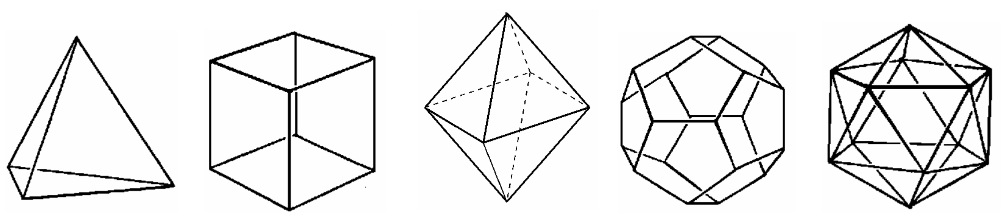
Components:

- **Faces** are the polygons that bound the polyhedron
- **Edges** are the lines where two faces join.
- **Vertices** are the points where three or more edges meet
- **Diagonal** is a segment that joins two non-consecutive vertices
- **Dihedron angle** is the angle between two faces

8.1 Regular polyhedrons.

The **regular polyhedrons** have all their faces formed by identical regular polygons. They are:

Tetrahedron: Four equal faces, each of them is a equilateral triangle
Cube: Six squares
Octahedron: Eight equilateral triangles
Dodecahedron: twelve regular pentagons
Icosahedron: Twenty equilateral triangles

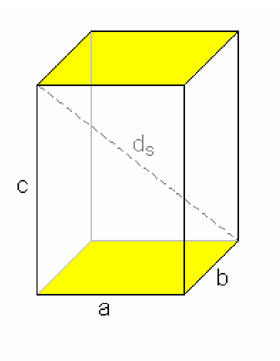


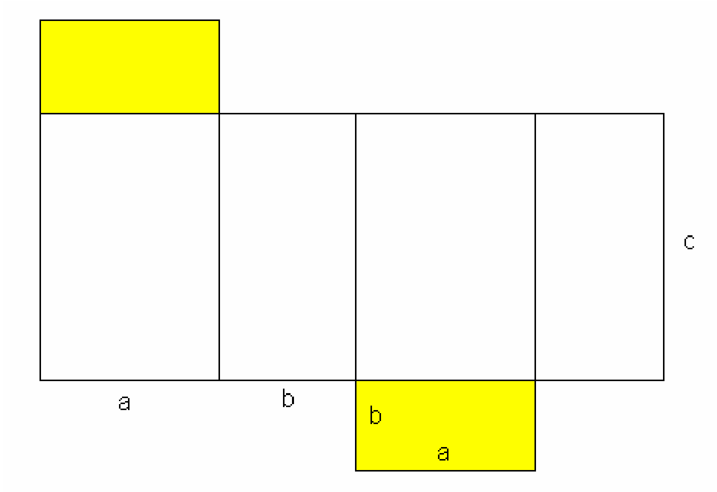
Tetrahedron Cube Octahedron Dodecahedron Icosahedron

8.2 Cuboids

A cuboid is a geometric objet with faces that are rectangles (in some cases squares). The special case in which all the faces are squares is the **cube.**

They have 6 faces, 12 edges and 8 vertices





8.3 Prisms

A prism is a polyhedron with two equal and parallel faces that are polygons (bases) and the other faces are parallelograms.

The distance between the two bases is the **height** of the prism.

Prisms can be right prisms when the parallelogram faces are perpendicular to the bases, otherwise they are oblique prisms.

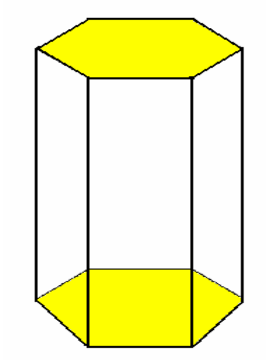
Depending on the polygons of the bases they can be:

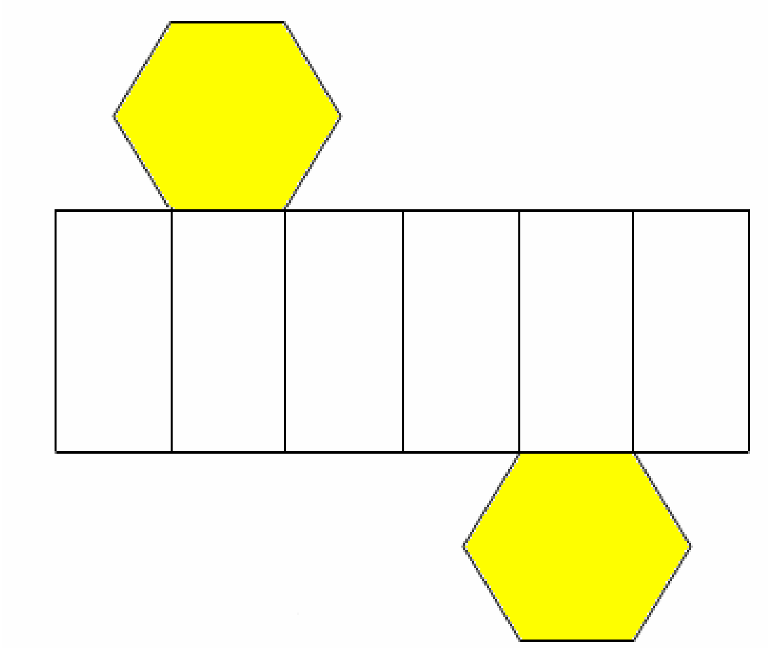
Triangular prism, with triangular bases

Square prism, with square bases (these are also called cuboids or parallelepipeds),

Pentagonal prism, when the bases are pentagons.

Hexagonal prism, etc.





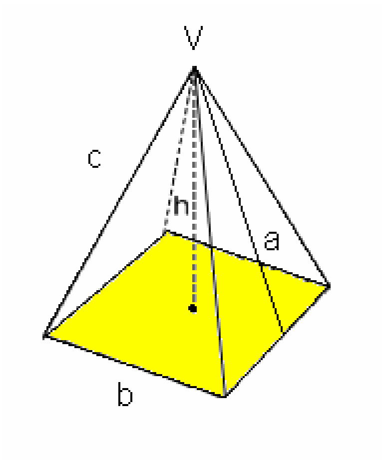
Hexagonal prism

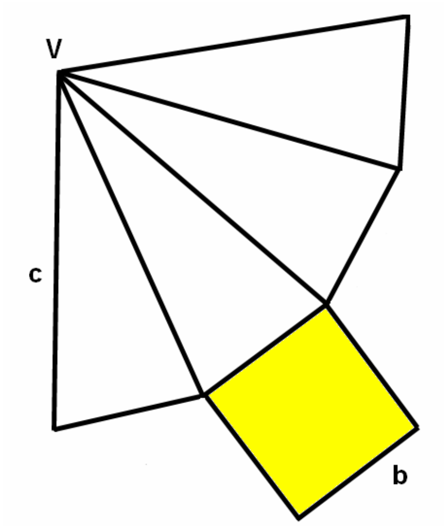
8.4 Pyramids

A pyramid is a polyhedron in which one of its faces is any polygon called the **base** and the other faces are triangles that join in a point that is the **apex.**

The height of a pyramid (h) is the distance from the base to the apex.

Like the prisms, pyramids can also be right or oblique and depending on the polygons of the base they can be: **triangular, square, pentagonal, hexagonal,** etc.





Square pyramid

8.5 Euler formula

The Euler formula relates the number of vertices *V*, edges *E*, and faces *F* of any polyhedron:

$F + V = E + 2$

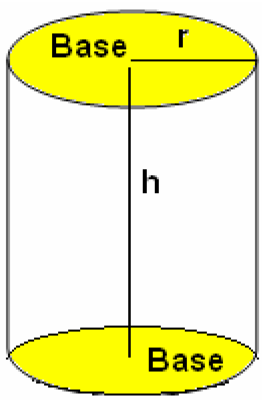
Exercise18. Complete this table and check in each case that the Euler formula comes true.

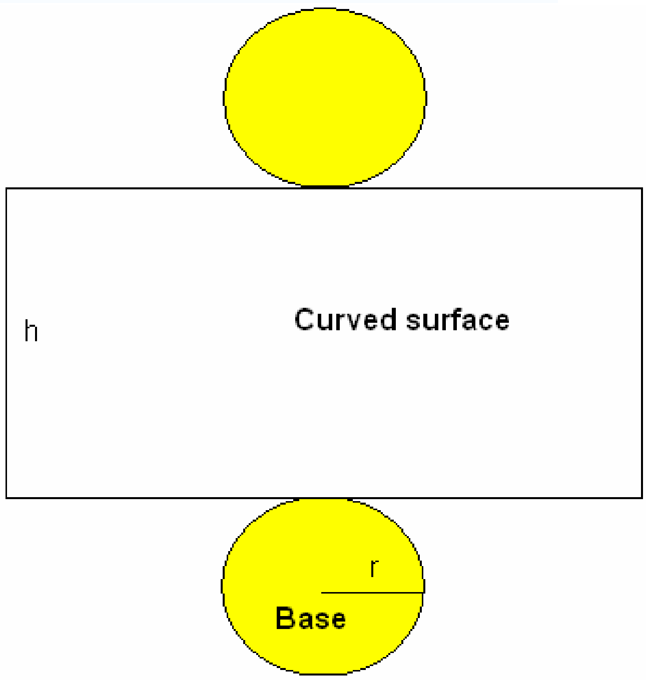
NAME OF THE SOLID	Number of faces	Number of vertices	Number of edges
Cuboid			
Tetrahedron			
Cube			
Octahedron			
Dodecahedron			
Triangular prism			
Pentagonal prism			
Hexagonal pyramid			
Octagonal pyramid			

9 Solids of revolution

9.1 Cylinders

A **cylinder** is a curvilinear geometric solid formed by a **curved surface** with all the points at a fixed distance from a straight line that is the **axis** of the cylinder and by two circles perpendicular to the axis that are the **bases**.

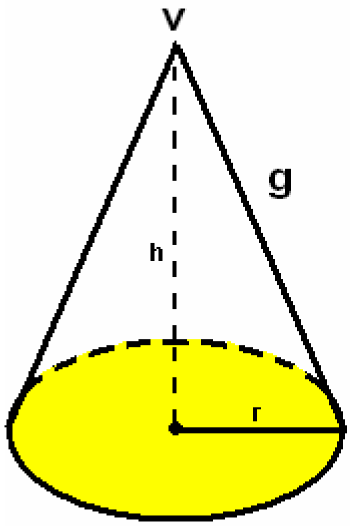


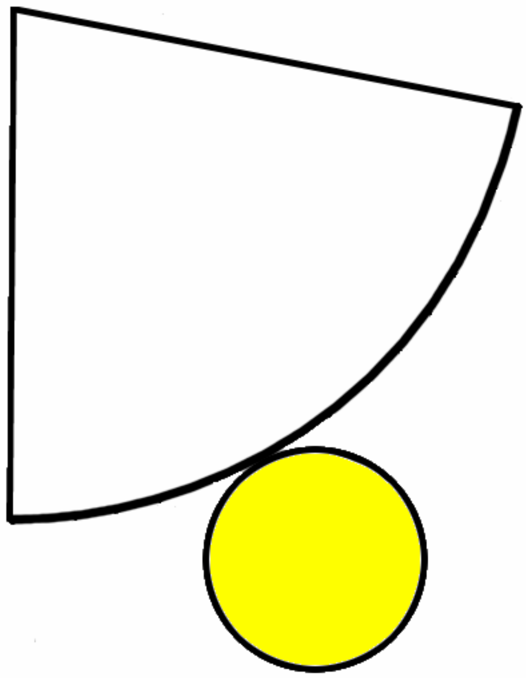


The height of a cylinder (h) is the distance from the base to the top.
The pattern of the curved surface when it is unrolled is a rectangle
The two bases are circles; the radius of the cylinder is the radii of the bases.

9.2 Cones

A cone is a solid bounded by a curved surface that has a common point (vertex), with a line that is the axis of the cone and a circle perpendicular to the axis that is called the base of the cone.





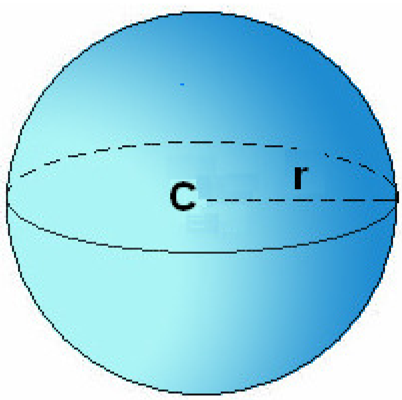
Vertex or apex is the top of the cone (**V**).

Generatrix of the cone is the straight line that joins the vertex with the circle of the base (**g**).

9.3 Sphere

A sphere is the solid bounded by a surface in which all points are at the same distance **r** from a fixed point that is the **centre** of the sphere **C**.

The distance from the centre to the surface of the sphere is called the radius of the sphere **r**



Solutions

Exercise 1

Triangle	By the sides	By the angles	Calculate
ABC	Scalene	Acute triangle	
DEF	Isosceles	Acute triangle	
HGI	Equilateral	Acute triangle	$\hat{H} = 60º$
JKL	Scalene	Right triangle	
MNO	Scalene	Obtuse triangle	$\hat{N} = 32º$
YAZ	Isosceles	Right triangle	$\hat{Y} = 45º$ $\hat{Z} = 45º$
BCD	Isosceles	Acute triangle	$\hat{C} = 50º$ $\hat{D} = 50º$
QPR	Isosceles	Obtuse triangle	$\hat{P} = 120º$
SUT	Isosceles	Acute triangle	
XVW	Scalene	Acute triangle	$\hat{X} = 75º$
MKL	Isosceles	Obtuse triangle	$\hat{L} = 100º$
NPO	Isosceles	Acute triangle	$\hat{O} = 75º$
GEF	Scalene	Right triangle	$\hat{F} = 90º$
QSR	Equilateral	Acute triangle	$\hat{R} = 60º$
UTV	Equilateral	Acute triangle	$\hat{U} = \hat{T} = \hat{V} = 60º$
HIJ	Scalene	Right triangle	$\hat{I} = 70º$

All exercises from 2 to 6 have to be drawn in class.
Exercise 7 a) c = 5.66 cm, b) a = 5 cm, c) b = 9.07 cm, d) b = 0.21 cm.
Exercise 8 side 4.47 cm, perimeter 16.9 cm.
Exercise 9 diagonal 9.9 cm.
Exercise 10 perimeter 16.12 cm.
Exercise 11 perimeter 21.2 cm
Exercise 12 a) 4, b) 2, c) 2, d)1 (if It is isosceles), e) 5 (regular pentagon)f) 6 (regular hexagon).
Exercise 13 polygon 1. $\angle ADC = 27º$; polygon 2. $\angle DAC = 29º$ $D = 3.31\text{cm}$; polygon 3. $\angle BAD = \angle BCD = 56º$, $h = 3.46\text{cm}$, $DC = 8\text{cm}$; polygon 4. $\hat{A} = 128º$, $\hat{B} = 37º$; polygon 5. $AB = 3.6\text{cm}$, $CD = 3.16\text{cm}$, $\angle ABC = 129º$, $\hat{D} = 64º$.
Exercise 14 2.24 cm.
Exercise 15 apothem 2.6 cm, central angle 60º.

Exercise 16 It should be drawn during class.
Exercise 17 b) $d = 3.57\,cm$.
Exercise18.

NAME	faces	vertices	edges
Cuboid	6	8	12
Tetrahedron	4	4	6
Cube	6	8	12
Octahedron	8	6	12
Dodecahedron	12	20	30
Triangular prism	5	6	9
Pentagonal prism	7	10	15
Hexagonal pyramid	8	12	18
Octagonal pyramid	9	9	16

In All solids faces + vertices = edges + 2 comes to be true.